P.G. Semester-II Examination, 2023

MATHEMATICS

Course ID : 22153 **Course Code : MATH203C**

Course Title : Calculus of Several Variables & Differential Geometry of Curves and Surfaces

Time : 2 Hours Full Marks : 40

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and symbols have their usual meaning.

GROUP–A

(Calculus of Several Variables)

Answer any **three** of the following questions: 8×3=24

1. i) If a function $f:\mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $c \in \mathbb{R}^n$, then show that *f* is continuous at *c*.

   ii) If a function $f:\mathbb{R}^n \to \mathbb{R}$ is differentiable at $c \in \mathbb{R}^n$, then show that

$$Df(c) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

   iii) Find the directional derivative of the function $f:\mathbb{R}^3 \to \mathbb{R}$, given by $f(x, y, z) = x^2 + y^2 - z^2$ in the direction (1, 1, 0) at (1, 0, 1). 2+3+3

2. i) If $f:\mathbb{C} \to \mathbb{C}$ is a complex differentiable function, then show that the corresponding real function $f:\mathbb{R}^2 \to \mathbb{R}^2$ is real differentiable. Is the converse true? Support your answer.

   ii) Let $f:\mathbb{R}^m \to \mathbb{R}^p$ and $g:\mathbb{R}^n \to \mathbb{R}^m$ be two vector functions. If *g* is differentiable at $a \in \mathbb{R}^n$ and *f* is differentiable at $g(a) \in \mathbb{R}^m$, then show that $(f \circ g):\mathbb{R}^n \to \mathbb{R}^p$ is differentiable at *a*.

(2+1)+5

3. i) State and prove the Taylor's theorem for several variables.

   ii) Suppose that $f:V \to \mathbb{R}$ is defined on an open set $V \subset \mathbb{R}^2$. If f_x, f_y and f_{xy} exist at every point of *V*, and f_{xy} is continuous at some point $(a, b) \in V$, then f_{yx} exist at (*a*, *b*) and $f_{yx} = f_{xy}$. 5+3

4. i) Let *V* be open in $\mathbb{R}^n$ and $f:V \to \mathbb{R}^n$ be C^1 on *V*. If $\det(Df(a)) \neq 0$ for some $a \in V$, then prove that there exists an open set *W* containing *a* such that *f* is 1-1 on *W*.

ii) Find the second and third derivatives of the function $f:\mathbb{R}^2 \to \mathbb{R}$, given by $f(x, y) = x^2y^2$ at (0, 1). 4+4

5. i) If f, g are Riemann integrable functions on a domain $\Omega \subset \mathbb{R}^n$ then show that

$$\int_\Omega (af + bg) = a\int_\Omega f + b\int_\Omega g,\ \forall a, b \in \mathbb{R}. \qquad 3$$

ii) State and prove Fubini's theorem for Riemann integrable functions. 5

GROUP–B

(Differential Geometry of Curves and Surfaces)

Answer any **two** of the following questions: 8×2=16

6. i) If A^i and B^j are the components of two contravariant vectors, then prove that their outer product is a tensor of type (2, 0). But the converse is not true.

ii) If a vector has contravariant components $(\ddot{x}, \ddot{y})$ in Cartesian coordinates, then find its components in polar coordinates.

iii) Find the torsion of the curve $\gamma(u) = 3(\cos u, \sin u, \cos 2u)$. 2+3+3

7. i) Find the signed curvature of the plane curve $y = \cosh x$.

ii) Find out a unit speed reparametrization of the space curve $\gamma(\theta) = (a\cos\theta, a\sin\theta, b\theta)$, where a and b are two constants.

iii) State and prove first Bianchi's identity. 2+3+3

8. i) Find the fields of three fundamental directions on the helix

$$\gamma(s) = \left(\frac{4}{5}\cos s, \frac{4}{5}\sin s, \frac{3}{5}s\right).$$

ii) Show that area of a surface patch is invariant under reparametrization.

iii) Find the second fundamental form of the surface of revolution

$$\sigma(u, v) = (f(u)\cos v, f(u)\sin v, g(u)),\ f(u) > 0, \forall u.$$

2+3+3

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